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A charged particle with magnetic moment in a medium with a disclination

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Abstract
Using a description of defects in solids in terms of three-dimensional gravity, we will consider a charged quantum particle with spin-$\frac{1}{2}$ in the field of a finite magnetic flux in a space with a disclination. It will be demonstrated that the change in topology caused by the defect produces modifications in the energy spectrum of a charged particle which in this case depends on the magnetic flux and on the global aspects.

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1. Introduction

Quantum effects on particles moving in crystalline media with topological defects have attracted considerable attention since the early 1950s [1]. More recently, a geometrical approach has been used [2, 3] to study such effects.

The advantage of a geometric description of defects in solids is twofold. Firstly, in contrast to the ordinary elasticity theory, this approach provides an adequate language for continuous defects. Secondly, the mighty mathematical machinery of differential geometry clarifies and simplifies any calculations.

In the continuum approximation that we use, the core is shrunk to a singularity. Although not very realistic, this model is very useful for showing the appearance of global phenomena related more to the topology than to the local geometry induced by the defects. These defects, although formed during the phase transition involving symmetry breaking, can be conceptually generated by a 'cut-and-glue' process, known in the literature as the Volterra process [4]. Among the variety of line defects generated in this process we have chosen to work with disclinations for their simplicity. It is important to stress the fact that real three-dimensional solids cannot have isolated disclinations because of the tremendous cost in the elastic energy associated in their formation. They are nevertheless common in two-dimensional systems appearing, for example, in graphite and liquid crystals [5, 6].

In a recent work, one of the authors of [7] demonstrated that the charge in topology caused by the defect produces on a free particle energy spectrum an analogue of the bound-state Aharonov–Bohm effect in condensed matter. Since the investigation concerns a scalar particle the inclusion of the spin is natural. In this paper we will consider a charged-quantum-mechanical particle with a spin-$\frac{1}{2}$ in a magnetic-string field in a disclination space. We want to ascertain whether a single disclination causes a change in the bound-state spectrum of a charged-quantum-mechanical particle with spin-$\frac{1}{2}$.

We use Katanaev and Volovich's approach [8] which translates the theory of defects in solids to the language of three-dimensional gravitation. A disclination is viewed [9] as the analogue of a cosmic string [10]: a topological defect carrying curvature but not torsion (in contrast to dislocations that carry torsion but not curvature). The disclination is obtained conceptually by either removing (positive-curvature disclination) or inserting (negative-curvature disclination) a wedge of dihedral angle $2\pi|\alpha - 1|$ such that the total angle around the z-axis is $2\pi\alpha$ instead of 2π. Therefore, $\alpha < 1$ corresponds to a angle deficit or positive disclination, and $\alpha > 1$ corresponds to an excess angle or negative disclination. The resulting space has a null curvature tensor everywhere except at the defect where it has a two-dimensional δ-function singularity. The space resulting is usually named 'conic'.

2. Bound-state energy

We are interested in the study of bound states in a space with a disclination submitted to a magnetic string potential

$$\vec{A} = \frac{\Phi}{2\pi\rho}\hat{\theta} \tag{1}$$

with the flux Φ. The interaction of a charged particle with this potential is described by the minimal coupling

$$\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}. \tag{2}$$

In a recent paper [5], we have studied a free particle in a disclination space. Whilst the initial investigation concerns a scalar particle, the inclusion of the spin is natural. In the case of a particle with spin s, there is an additional term with respect to (2); the interaction of its magnetic moment

$$\mu = \frac{g\mu_B}{\hbar}s \tag{3}$$

where $\mu_B = e\hbar/2mc$ and g is its gyromagnetic ratio.

The metric in the space with a disclination is characterized by

$$\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{d}\rho^2 + \alpha^2\rho^2\,\mathrm{d}\theta^2. \tag{4}$$

In order to simplify this demonstration, we consider a non-relativistic Hamiltonian operator which

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + \mu H. \tag{5}$$

The second term in (5) is the contribution due to the interaction spin magnetic moment with μ being given by (3). Note that $\vec{p} = -\hbar\mathrm{i}\vec{\nabla}_\alpha$, wherein ∇_α is the gradient operator in the background space described by (4).

The model which we have used is

$$\vec{H}(\rho) = \frac{\Phi}{2\pi R}\delta(\rho - R)\hat{z}. \tag{6}$$

From (6) the vector potential can be written as

$$\vec{A} = \frac{\Phi}{2\pi\rho} a(\rho)\hat{\theta}. \tag{7}$$

Due to spin conservation, the magnetic interaction (5) can be replaced by

$$\pm g\mu_B H(\rho) \tag{8}$$

wherein $\pm$ corresponds to the spin projection on the flux line. From here, we have restricted ourselves to the minus sign in which the magnetic moment leads to a binding force. Furthermore, we choose $\Phi > 0$; for $\Phi < 0$ the spin direction must be reversed.

In order to compute the modification of the energy spectrum due to the disclination, one needs to write the Schrödinger equation for the space endowed with the metric (4)

$$-\left[\nabla_\alpha^2 + \frac{2ei}{\hbar c}\frac{\Phi}{2\pi\alpha\rho^2} a(\rho)\frac{\mathrm{d}}{\mathrm{d}\theta} + \frac{e^2\Phi^2}{4\pi^2c^2\rho^2} a^2(\rho) + \frac{2m}{\hbar^2}\frac{g\mu_B H(\rho)}{2}\right]\Psi = \varepsilon\Psi \tag{9}$$

wherein

$$\nabla_\alpha^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\alpha^2\rho^2}\frac{\partial^2}{\partial\theta^2} \tag{10}$$

is the Laplace–Beltrami operator in the background space described by (4) and $\varepsilon = 2mE/\hbar^2$.

Using the ansatz

$$\Psi(\rho,\theta,z) = \mathrm{e}^{\mathrm{i}\ell\theta}\mathrm{e}^{\mathrm{i}\kappa z}\Psi(\rho) \tag{11}$$

and substituting in (9) (for simplicity we set $\kappa = 0$) we obtain

$$\left[-\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right) + \frac{(\beta - \delta' a(\rho))^2}{\rho^2} - \frac{g\,\delta'}{2\rho}\frac{\mathrm{d}a(\rho)}{\mathrm{d}\rho}\right]\Psi(\rho) = \varepsilon\Psi \tag{12}$$

where

$$\beta = \frac{\ell}{\alpha} \qquad \mu_B = \frac{e\hbar}{2mc} \qquad \text{and} \qquad \delta' = \frac{\Phi e}{hc}. \tag{13}$$

Substituting (6) in (12) gives

$$\left[-\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right) + \frac{(\beta - \delta'\Theta(R-\rho))^2}{\rho^2} - \frac{g}{2}\delta'\frac{\delta(\rho - R)}{R}\right]\Psi(\rho) = \varepsilon\Psi(\rho) \tag{14}$$

where $\Theta(R-\rho)$ is the Heaviside theta function.

The regularization of the δ-function interaction can be done by many different models for a finite flux tube. We used here the same model as in [11], namely we have written down the wavefunctions inside the tube and connected them with the outside function.

In the inside case ($\rho < R$) the expression (14) can be written as

$$\left[\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right) + \left(\varepsilon - \frac{\beta^2}{\rho^2}\right)\right]\Psi(\rho) = 0 \tag{15}$$

whose solution is

$$\Psi_{|\beta|} = \tau J_{|\beta|}(\sqrt{\varepsilon}\rho) \tag{16}$$

where $J_{|\beta|}(\sqrt{\varepsilon}\rho)$ is the Bessel function and β is given in (13).

Alternatively for the outside case ($\rho > R$) we have

$$\Psi_\beta = J_{|\beta-\delta'|}(\sqrt{\varepsilon}\rho) + C_\beta(\sqrt{\varepsilon})H_{|\beta-\delta'|}(\sqrt{\varepsilon}\rho) \tag{17}$$

where $H_{|\beta-\delta'|}(\sqrt{\varepsilon}\rho)$ are the Hankel functions. This solution is identical to that described in [11].

We choose to work with the Hankel function rather than the Neumann one, due to the asymptotic behaviour, $e^{ik\rho}$, for $\rho \to \infty$.

In the case considered here since the δ-function is moved away from the origin it can be treated as an one-dimensional case and can be substituted by the known boundary conditions as follows:

$$\rho\frac{\partial\Psi_{|\beta|}}{\partial\rho} = -\frac{g}{2}\delta'\Psi_\beta(\rho)_{|_R} \tag{18}$$

obtained from (14). In addition the solution must also be determined by the condition that

$$R_{\text{int}} = R_{\text{ext}}. \tag{19}$$

Note that α and ℓ must be integers and from equation (13) we see that β is not restricted to integer values.

For the inside case ($\rho < R$) using (18) we have [11]

$$R_{\text{int}} = -\frac{1}{2}g\delta' + \sqrt{\varepsilon}R\frac{J'_{|\beta|}(\sqrt{\varepsilon}R)}{J_{|\beta|}(\sqrt{\varepsilon}R)} \tag{20}$$

for $\varepsilon > 0$. When $\varepsilon < 0$ we obtain

$$R_{\text{int}} = -\frac{1}{2}g\delta' + \sqrt{\varepsilon}R\frac{I'_{|\beta|}(\sqrt{-\varepsilon}R)}{I_{|\beta|}(\sqrt{-\varepsilon}R)} \tag{21}$$

with $I_{|\beta|}(\sqrt{\varepsilon}R)$ denoting the modified Bessel function of the first kind and order β. In the outside case ($\rho > R$) we can write [11]

$$R_{\text{ext}} = -\frac{1}{2}g\delta' + \sqrt{\varepsilon}R\frac{J'_{|\beta-\delta'|}(\sqrt{\varepsilon}R) + B_\beta(\sqrt{\varepsilon})H'^{(1)}_{|\beta-\delta'|}(\sqrt{\varepsilon}R)}{J_{|\beta-\delta'|}(\sqrt{\varepsilon}) + B_\beta(\sqrt{\varepsilon})H^{(1)}_{|\beta-\delta'|}(\sqrt{\varepsilon}R)} \tag{22}$$

where $H^{(1)}_{|\beta-\delta'|}$ is the Hankel function of the first kind, for $\varepsilon > 0$. On the other hand, in the case of $\varepsilon < 0$ we obtain [11]

$$R_{\text{ext}} = -\frac{1}{2}g\delta' + \sqrt{-\varepsilon}R\frac{K'_{|\beta-\delta'|}(\sqrt{-\varepsilon}R)}{K_{|\beta-\delta'|}(\sqrt{-\varepsilon}R)} \tag{23}$$

where $K_{|\beta-\delta'|}(\sqrt{-\varepsilon}\,R)$ is the modified Bessel functions of the second kind and $B_\beta(\sqrt{\varepsilon}\,)$ is a constant.

We shall now consider the bound-states solutions, namely, $\varepsilon < 0$.

Using (21) and (23) we can easily show that the solution for the bound-state, in general, from (19) is

$$y = f(\delta', g, \beta) \tag{24}$$

where

$$y = \kappa_\beta R = \sqrt{-\varepsilon}R. \tag{25}$$

Note that because β depends on α, the binding energy are influenced by α even though the space is locally flat. Namely the particle depends on the global-aspect features of this space.

In the limit $R \to 0$ with all the other parameters fixed and the flux string coinciding with the defect, the bound state tends to infinity as can be seen from (25). This indicates that it is not a physical limit at least in the model used here for $g > 2$. The reason for this is that the bound-state energies k_m enter the defining equation (25) multiplied by the flux-tube radius, R; this is also implied by the general-dimensional considerations.

One possibility is for the gyromagnetic ratio g in equation (14) to change as $R \to 0$. A more detailed study can be seen in [11], since our aim in this work is just to study how the magnetic effect combines to affect energy levels.

3. Conclusions

We have demonstrated that the binding energy of a particle is affected by the existence of disclination. It is important to remark here that the region outside the defect has zero curvature, while the defect region has all the curvature. Also, in the case where the magnetic flux coincides with the defect ($R \to 0$) (the well known Aharonov–Bohm [7]), the bound-state energies tend to be infinite so that this limit is not physical. In summary, we have demonstrated that the particle bound state depends on the magnetic flux as well as on the global features of this space. Namely, there is a combination of topological and electromagnetic effects.

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